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Solution of an Optimal Control Problem with Vector Control using Relaxation Method

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Abstract

An optimal control problem with a linear state equation and vector control with the minimization of a quadratic criterion is considered. Furthermore, the terminal state is fixed with fixed horizon. The relaxation method is proposed coupled with the shooting technique to solve this problem. We rely on shooting method to find initial condition of adjoint vector $p(0)$. The convergence of the numerical procedure is presented where numerical experiments are also presented. We solve the problem analytically then we compare the results.

Keywords: *Optimal control, Shooting method, Relaxation method.*

1 Introduction

The theory of optimal control has been well developed for over forty years. With the advances of computer technique, optimal control is now widely used in multi-disciplinary applications such as biological systems, communication networks and socio-economic systems etc. As a result, more and more people will benefit greatly by learning to solve the optimal control problems numerically. Realizing such growing needs, books on optimal control put more weight on numerical methods. In retrospect, [11] was the first and the "classic" book for studying the theory as well as many interesting cases (time-optimal, fuel-optimal and linear quadratic regulator(LQR) problems). Necessary conditions for various systems were derived and explicit solutions were given when possible.

In this work, we are interested in linear quadratic optimal control problems with a fixed terminal time. Moreover the state is subject to some constraints and a value of the final state is fixed and vector control. In order to use a numerical solution, we rewrite the equations of optimality from the Pontryagin's minimum principle. Then we obtain a system in which, the differential equation describing the state is equipped with an initial and final conditions. Note also that, the costate equation derived from the Pontryagin's principle is equipped with no initial condition or terminal one to be used algorithmically. In order to determine the initial condition of the costate we use in this study, the shooting method [5] coupled with the relaxation method (see [1] and [3]). Under suitable assumptions, we analyze the convergence of the considered iterative method.

This study is organized as follows. Section 2 defines the problem and presents the shooting method. The Section 3 is devoted to the description of the relaxation method coupled with the shooting one. The convergence of the proposed method is presented in section 4, while Section 5 contains the results of the numerical experiments.

2 Statement of the Problem

Let us consider the following problem:

$$\begin{cases} Found \quad u \in \mathcal{U}_{ad} \quad such\ that, \\ \\ J(u) \leq J(v), \ \forall\ v \in \mathcal{U}_{ad}, \end{cases}$$

Where

$$J(u) = \frac{1}{2}\int_0^T ((x - x_d)^2 + ku^2)dt;$$

x_d corresponding at desired state, u is the control and k control the comparative weight given to the two components of the cost function, i.e., the values of k are chosen in order to grant more weight to precision or energy expenditure to the final state x_T fixed and minimizing the cost functional, and $\mathcal{U}_{ad}$ is the closed convex domain.

Under the following constraints:

$$\begin{cases} \dot{x}_1 = -bx_1 + ax_2 + u_1, \\ \dot{x}_2 = ax_1 - bx_2 + u_2, \\ x(0) = x_0, \\ \kappa x(T) = g, \end{cases}$$

With $G = (1, 2),\ \gamma = 3,\ \kappa = (1, 0),\ g = 2,\ T = 2,\ a > 0,\ b > 0.$

Otherwise, We can reformulate the problem as follows:

$$\begin{cases} \dot{x}_1 = -bx_1 + ax_2 + u_1, \\ \dot{x}_2 = ax_1 - bx_2 + u_2, \\ x(0) = x_0, \\ \\ x_1(T) = 2, \end{cases}$$

The matrix form of the state system is written as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -b & a \\ a & -b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

with

$$J(u) = \frac{1}{2}\int_0^T [(x_1 - x_{1d}, x_2 - x_{2d})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 - x_{1d} \\ x_2 - x_{2d} \end{pmatrix} + k(u_1, u_2)^t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}]dt.$$

If we solve the previous problem by a numerical procedure, we have then to solve an algebraic-differential systemØš On one hand, the state equation describing the physical system is provided with an initial condition x_0 and the final condition x_f. On the other hand, the second equation corresponding to the costate one is formed with no initial condition and with no terminal condition. We will therefore use the shooting method to compute the value of $p(0)$.

2.1 Shooting method

The shooting method is used to obtain the value of $p(0)$ necessary to the solution of the problem characterized by the Pontryaguin principle. If it is possible from the condition of minimization of the Hamiltonian to express the extremal control with respect to $(x(t), p(t))$, then the extremal system is a differential system of the form $\dot{z}(t) = F(t, z(t))$, where $z(t) = (x(t), p(t))$. With a numerical integrator beginning from z_0 we obtain: $\tilde{z}_i^{z_0} \sim z(t_i)$, where the t_i are the discretized time generated by the numerical integrator. Note that in $z_0 = (x_0, p_0)$, the value of x_0 is given by the initial condition of the problem and p_0 denotes the value of $p(0)$. Therefore, for different values of p_0, we will obtain the corresponding values of $\tilde{z}_i^{z_0}$. We are interested in the value of $\tilde{z}_N^{z_0} \sim z(T)$, at the final time, where N is such that $T = N.dt$, dt being the time step, and $\tilde{z}_N^{z_0} = (\tilde{x}_N^{z_0}, \tilde{p}_N^{z_0})$. In our case only $\widehat{x}_N^{z_0}$ are significant. Since the numerical results only depend on p_0 they are denoted in the sequel $\tilde{x}_N^{p_0}$. We define

the mapping G as an implicit function which gives p_0 by using a numerical integrator such that we obtain $\tilde{x}_N^{p_0} - x_T = 0$. Let

$$G : \mathbb{R}^n \to \mathbb{R}^n \ \ \textit{such that} \ \ G(p_0) = \tilde{x}_N^{p_0} - x_T.$$

Using G we define a nonlinear implicit system of n equations with n unknowns such that:

$$G(p_0) = 0. \tag{1}$$

In order to solve (1), we will use the Newton's method. The principle of the Newton's method is described as follows: in the k^{th} step, let p_0^k be an approximation of the zero p_0 of G; therefore p_0 can be written $p_0 = p_0^k + \Delta p_0^k$ and then:

$$0 = G(p_0) = G(p_0^k + \Delta p_0^k) = G(p_0^k) + \frac{\partial G}{\partial p_0}(p_0^k).(p_0 - p_0^k) + o(p_0 - p_0^k),$$

which leads to the solution of the following linear system:

$$\frac{\partial G}{\partial p_0}(p_0^k).(p_0 - p_0^k) = -G(p_0^k),$$

where $\frac{\partial G}{\partial p_0}(p_0^k)$ is the Jacobian matrix of the application $p_0 \to G(p_0)$ computed when $p_0 = p_0^k$; note that the mapping $p_0 \to G(p_0)$ is not explicitly known but is known numerically. So we will use a method of numerical derivation based on the finite difference method. To avoid the computation of $\frac{\partial G}{\partial p_0}(p_0^k)$, it is sufficient to find an approximation of $\frac{\partial G}{\partial p_0}(p_0^k)$. According to [4], we have two typical finite difference approximations which are frequently used:

$$\frac{\partial G_i}{\partial p_{0j}}(p_0^k) \approx \frac{1}{h_{ij}}[G_i(p_0 + \sum_{k=1}^{j} h_{ik}e^k) - G_i(p_0 + \sum_{k=1}^{j-1} h_{ik}e^k)],$$

or

$$\frac{\partial G_i}{\partial p_{0j}}(p_0^k) \approx \frac{1}{h_{ij}}[G_i(p_0 + h_{ij}e^j) - G_i(p_0)],$$

where the h_{ij} are the given discretization step parameters of the i^{th} equation with respect to the j^{th} variable and e^k are the k^{th} vector of the canonical basis; note that, classically, we can always choose the values of h_{ij} equal to each other at a common value h. Let $\Delta_{ij}(p, h)$ be a finite difference approximation, then we have:

$$\lim_{h \to 0} \Delta_{ij}(p_0, h) = \frac{\partial G_i}{\partial p_{0j}}(p_0), \quad i, j = 1, ..., n.$$

Let

$$J(p, h) = (\Delta_{ij}(p_0, h)),$$

which is an approximation of the Jacobian matrix; then the approximate Newton's method can be written as follows:

$$p_0^{k+1} = p_0^k - J(p_0^k, h^k)^{-1}.G(p_0^k), \quad k = 0, 1, \dots,$$

The problem of convergence of this iterative process is ensured by using a result of the book of Ortega and Rheinbold [4]; indeed, if the discretization step h_{ij} are small enough and tend to zero then the convergence is ensured.

3 Numerical Algorithm

To solve the problem, with and without constraints, we perform the coupling of the relaxation method (see [1] and [3]) with the shooting method [5], the latter method being intended to calculate $p(0)$ necessary to solve the differential-algebraic system obtained by applying the Pontryagin minimum principle. The steps in this numerical method are summarized below:

In the unconstrained case note that $\bar{x} \equiv x^{(r)}$.

Remark 1 *Steps (2), (2) and (3)correspond to the relaxation method while the following steps correspond to the implementation of the shooting method.*

Definition 1 *An invertible matrix $\bar{A}$ is an M-matrix if $\bar{A}^{-1} \geq 0$ and $\bar{a}_{ij} \leq 0$ for $i \neq j$.*

Remark 2 *M-matrices have many main properties, in particular the spectral radius of the associated Jacobi matrix $J = I - \bar{D}^{-1}.\bar{A}$, where $\bar{D}$ is the diagonal of $\bar{A}$, is lower than one; this property will be used in the sequel.*

Algorithm 1 Numerical Algorithm

$\diamondsuit$ **Initial Approximation**

Control u^0 for $t \in [0, T]$, and of the costate variable $p^0(0)$,

$\diamond-$ $r \leftarrow 0$ (where r is the number of iteration),

$\diamondsuit$ **If** $\mid u^{(r+1)} - u^{(r)} \mid > \epsilon$ (%where ϵ defines the convergence threshold) **do**:

- Determine the state variable $x^{(r)}$, by integration of the state equation:

$$\begin{cases} \frac{d\bar{x}}{dt} = A\bar{x} + Bu^{(r)}, & 0 < t \leq T, \\ \bar{x}(0) = x_0, \end{cases}$$

$$and x^{(r)} = Proj(\bar{x}),$$

where $Proj(.)$ is the projected operator,

-and find the costate variable $p^{(r)}$ by solving:

$$\begin{cases} -\frac{dp^{(r)}}{dt} = \frac{\partial H}{\partial x^{(r)}}, \\ p^{(r)}(0), \end{cases} \tag{2}$$

where $p^{(r)}(0)$ is computed by the shooting method,

$-$ Determine the control $u^{(r+1)}$:

$$u^{(r+1)} \leftarrow (u^{(r)} - \frac{1}{k} B^t p^{(r)}), \tag{3}$$

$\diamond-$ Else $\mid u^{(r+1)} - u^{(r)} \mid < \epsilon$,

- Determine the shooting function:

$$G(p) = x^{(r)}(T) - x_f,$$

- Solve the shooting equation by the Newton method and find the new value of $p(0)$:

$$p^{(r+1)}(0) \leftarrow p^{(r)}(0) + correction,$$

- $r \leftarrow r + 1$.

§**End**

Proposition 1 *If the following conditions are satisfied:*
- $\bar{A}$ *is an M-matrix*
- $k \geq k_0 > 0$
- $p^2(0) - p^2(T) > 0,$

then the algorithm for computing the numerical optimal control law, by the relaxation method coupled with the shooting method, converges whatever be the initial value of u^0.

4 Numerical Experiments

The Hamiltonian of the system is given by:

$$\begin{aligned} H(x(t),p(t),u(t),t) &= \frac{1}{2}[(x_1-x_{1d})^2+(x_2-x_{2d})^2+k(u_1^2+u_2^2)] \\ &+ (p_1(t),p_2(t))\begin{pmatrix} -b & a \\ a & -b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (p_1(t),p_2(t))\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \frac{1}{2}[(x_1-x_{1d})^2+(x_2-x_{2d})^2+k(u_1^2+u_2^2)] + p_1(t)(-bx_1+ax_2+u_1) \\ &+ p_2(t)((ax_1-bx_2+u_2)+p_1(t)u_1+p_2(t)u_2. \end{aligned}$$

On the state the optimality equations can be written as follows:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{dx_1}{dt} = \frac{\partial H}{dx_1} = -bx_1+ax_2+u_1, \\ \frac{dx_2}{dt} = \frac{\partial H}{dx_2} = ax_1-bx_2+u_2, \\ x(0)=x_0, \end{array} \right. \\ \left\{ \begin{array}{ll} -\frac{dp_1}{dt} = \frac{\partial H}{dx_1} = -bp_1+ap_2+x_1-x_{1d}, & p_1(T)=\alpha_1 \\ -\frac{dp_2}{dt} = \frac{\partial H}{dx_2} = ap_1-bp_2+x_2-x_{2d}, & p_2(T)=\alpha_2 \end{array} \right. \\ \frac{\partial H}{\partial u_1} = 0 = ku_1+p_1(t). \\ \frac{\partial H}{\partial u_2} = 0 = ku_2+p_2(t). \end{array} \right.$$

where $\alpha_i, i=\overline{1,2}$ is found below. Adjunct on the previous system transversality condition:
$\exists\ k_1 \neq 0$ such that:

$$\varphi(0,x_0,T,x(T),k_0,k_1) = (\psi_1(T,x(T)|k_1),$$

with

$$\psi_1(T,x(T)) = \kappa x(T) - g,$$

$$\varphi(0,x_0,T,x(T),k_1) = (\kappa x(T) - g|k_1).$$

$$p(T) = -\frac{\partial\varphi}{\partial x(T)}(0,x_0,T,x(T),k_1) = -\kappa^t k_1,$$

$$(p_1(T),p_2(T)) = -(1,0)k_1 = (-k_1,0).$$

After, the values of $\alpha_i, i=\overline{1,2}$ are defined, Hamilton-Pontriaguine equation's are written as follows:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \dot{x}_1 = -bx_1+ax_2+u_1, \\ \dot{x}_2 = ax_1-bx_2+u_2, \\ x(0)=x_0, \end{array} \right. \\ \left\{ \begin{array}{ll} -\dot{p}_1 = -bp_1+ap_2+x_1-x_{1d}, & p_1(T)=-k_1 \\ -\dot{p}_2 = ap_1-bp_2+x_2-x_{2d}, & p_2(T)=0 \end{array} \right. \\ 0 = ku_1+p_1(t) \\ 0 = ku_2+p_2(t) \end{array} \right.$$

Then, the matrix form is given as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ -\dot{p}_1 \\ -\dot{p}_2 \\ 0 \\ 0 \\ p_1(T) \end{pmatrix} = \begin{pmatrix} -b & a & 0 & 0 & 0 & 0 & 0 \\ a & -b & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -b & a & 0 & 0 & 0 \\ 0 & -1 & a & -b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \\ u_1 \\ u_2 \\ k_1 \end{pmatrix}$$

We solve iteratively the following problem of the fixed point:

$$\begin{cases} \dot{x}_1 = -bx_1 + ax_2 + u_1, \\ \dot{x}_2 = ax_1 - bx_2 + u_2, \\ x_1(0) = 0, x_2(0) = 0, \\ \dot{p}_1 = bp_1 - ap_2 - x_1 + x_{1d}, \qquad p_1(T) = -k_1 \\ \dot{p}_2 = -ap_1 + bp_2 - x_2 + x_{2d}, \quad p_2(T) = 0, \\ u_1 = -\frac{p_1(t)}{k}, \\ u_2 = -\frac{p_2(t)}{k}, \end{cases}$$

5 Analytical Solution

Let us derivation method at the level equation. We obtain:

$$\dot{p}_1 = bp_1 - ap_2 - x_1 + x_{1d},$$

$$\ddot{p}_1 = b\dot{p}_1 - a\dot{p}_2 - \dot{x}_1,$$

$$\ddot{p}_1 = b(bp_1 - ap_2 - x_1 + x_{1d}) - a(-ap_1 + bp_2 - x_2 + x_{2d}) - (-bx_1 + ax_2 - \frac{p_1}{k}),$$

$$\ddot{p}_1 = b^2p_1 - abp_2 - bx_1 + bx_{1d} + a^2p_1 - abp_2 + ax_2 - ax_{2d} + bx_1 - ax_2 + \frac{p_1}{k},$$

$$\ddot{p}_1 = (a^2 + b^2 + \frac{1}{k})p_1 - 2abp_2 + bx_{1d} - ax_{2d},$$

Then,

$$\ddot{p}_1 = (a^2 + b^2 + \frac{1}{k})p_1 - 2abp_2 + bx_{1d} - ax_{2d}.$$

In the same way:

$$\dot{p}_2 = -ap_1 + bp_2 - x_2 + x_{2d}$$

$$\ddot{p}_2 = -a\dot{p}_1 + b\dot{p}_2 - \dot{x}_2,$$

$$\ddot{p}_2 = -a(bp_1 - ap_2 - x_1 + x_{1d}) + b(-ap_1 + bp_2 - x_2 + x_{2d}) - (ax_1 - bx_2 - \frac{p_2}{k}),$$

$$\ddot{p}_2 = -abp_1 + a^2p_2 + ax_1 - ax_{1d} - abp_1 + b^2p_2 - bx_2 + bx_{2d} - ax_1 + bx_2 + \frac{p_2}{k},$$

$$\ddot{p}_2 = -2abp_1 + (a^2 + b^2 + \frac{1}{k})p_2 - ax_{1d} + bx_{2d},$$

Then

$$\ddot{p}_2 \quad = \quad (a^2 + b^2 + \frac{1}{k})p_2 - 2abp_1 - ax_{1d} + bx_{2d} \tag{4}$$

derive double the equation (4), we obtain:

$$p_1^{(4)} \quad = \quad (a^2 + b^2 + \frac{1}{k})\ddot{p}_1 - 2ab\ddot{p}_2,$$

$$p_1^{(4)} \quad = \quad (a^2 + b^2 + \frac{1}{k})\ddot{p}_1 - 2ab(-2abp_1 + (a^2 + b^2 + \frac{1}{k})p_2 - ax_{1d} + bx_{2d}),$$

$$p_1^{(4)} \quad = \quad (a^2 + b^2\frac{1}{k})\ddot{p}_1 + 4a^2b^2p_1 - 2ab(a^2 + b^2 + \frac{1}{k})p_2 + 2a^2bx_{1d} - 2ab^2x_{2d}),$$

Then,

$$p_1^{(4)} \quad = \quad (a^2 + b^2 + \frac{1}{k})\ddot{p}_1 + 4a^2b^2p_1 - 2ab(a^2 + b^2 + \frac{1}{k}) + 2a^2bx_{1d} - 2ab^2x_{2d},$$

(4) leads:

$$2abp_2 = (a^2 + b^2)p_1 - \ddot{p}_1 + bx_{1d} - ax_{2d}, \tag{5}$$

By injecting (5) in (5), we obtain:

$$p_1^{(4)} \quad = \quad (a^2 + b^2 + \frac{1}{k})\ddot{p}_1 + 4a^2b^2p_1 - (a^2 + b^2 + \frac{1}{k})((a^2 + b^2 + \frac{1}{k})p_1 - \ddot{p}_1 + bx_{1d} - ax_{2d}) + 2a^2bx_{1d} - 2ab^2x_{2d},$$

$$\begin{aligned} p_1^{(4)} \quad &= \quad (a^2 + b^2 + \frac{1}{k})\ddot{p}_1 + 4a^2b^2p_1 - ((a^2 + b^2)^2 + \frac{1}{k}(a^2 + b^2))p_1 + (a^2 + b^2 + \frac{1}{k}\ddot{p}_1 \\ &- \quad b(a^2 + b^2 + \frac{1}{k})x_{1d} + a(a^2 + b^2 + \frac{1}{k})x_{2d} + 2a^2bx_{1d} - 2ab^2x_{2d}, \end{aligned}$$

$$\begin{aligned} p_1^{(4)} \quad &= \quad (2a^2 + 2b^2 + \frac{2}{k})\ddot{p}_1 - ((a^2 + b^2 + \frac{1}{k})^2 - 4a^2b^2)p_1 + (-a^2b - b^3 - \frac{b}{k} + 2a^2b)x_{1d} \\ &+ \quad (a^3 + ab^2 + \frac{a}{k} - 2ab^2)x_{2d}, \end{aligned}$$

Then,

$$p_1^{(4)} - (2a^2 + 2b^2 + \frac{2}{k})\ddot{p}_1 - ((a^2 + b^2 + \frac{1}{k})^2 - 4a^2b^2)p_1 = (-b^3 - \frac{b}{k} + a^2b)x_{1d} + (a^3 - ab^2 + \frac{a}{k})x_{2d}, \tag{6}$$

The characteristic equation corresponding in equation (6) written as:

$$C^4 - 2(a^2 + b^2 + \frac{1}{k})C^2 - (-(a^2 + b^2 + \frac{1}{k})^2 + 4a^2b^2) = 0,$$

$$\Delta = 4a^2b^2$$

The root of the characteristic equation are given by:

$$C_1^2 = (a^2 + b^2 + \frac{1}{k}) - 2ab,$$

$$C_2^2 = (a^2 + b^2 + \frac{1}{k}) + 2ab.$$

Then

$$p_1(t) \quad = \quad \lambda e^{C_1t} + \beta e^{-C_1t} + \mu e^{C_2t} + \alpha e^{-C_2t} + \nu. \tag{7}$$

Determine ν, we have:

$$\dot{p}_1(t) = \lambda C_1 e^{C_1 t} - \beta C_1 e^{-C_1 t} + \mu C_2 e^{C_2 t} - \alpha C_2 e^{-C_2 t},$$

$$\ddot{p}_1(t) = \lambda C_1^2 e^{C_1 t} + \beta C_1^2 e^{-C_1 t} + \mu C_2^2 e^{C_2 t} + \alpha C_2^2 e^{-C_2 t},$$

$$p_1^{(3)}(t) = \lambda C_1^3 e^{C_1 t} - \beta C_1^3 e^{-C_1 t} + \mu C_2^3 e^{C_2 t} - \alpha C_2^3 e^{-C_2 t},$$

$$p_1^{(4)}(t) = \lambda C_1^4 e^{C_1 t} + \beta C_1^4 e^{-C_1 t} + \mu C_2^4 e^{C_2 t} + \alpha C_2^4 e^{-C_2 t}.$$

We substitute on (6), and we obtain:

$$\begin{aligned} & \lambda[C_1^4 - (-(a^2+b^2+\frac{1}{k})^2 + 4a^2b^2) - 2(a^2+b^2+\frac{1}{k})C_1^2]e^{C_1 t} + \beta[C_1^4 - (-(a^2+b^2+\frac{1}{k})^2 + 4a^2b^2) \\ - & 2(a^2+b^2+\frac{1}{k})C_1^2]e^{-C_1 t} + \mu[C_2^4 - (-(a^2+b^2+\frac{1}{k})^2 + 4a^2b^2) - 2(a^2+b^2+\frac{1}{k})C_2^2]e^{C_2 t} + \alpha[C_2^4 \\ - & (-(a^2+b^2+\frac{1}{k})^2 + 4a^2b^2) - 2(a^2+b^2+\frac{1}{k})C_2^2]e^{-C_2 t} + (-(a^2+b^2+\frac{1}{k})^2 + 4a^2b^2)\nu \\ = & (-b^3 - \frac{b}{k} + a^2 b)x_{1d} + (a^3 - ab^2 + \frac{a}{k})x_{2d}, \end{aligned}$$

By identification, we obtain:

$$\nu = \frac{(-b^3 - \frac{b}{k} + a^2 b)x_{1d} + (a^3 - ab^2 + \frac{a}{k})x_{2d}}{4a^2b^2 - (a^2+b^2+\frac{1}{k})^2}. \tag{8}$$

We deduce $p_2(t)$ of (5), we obtain:

$$p_2(t) = \frac{1}{2ab}[(a^2+b^2+\frac{1}{k})p_1 - \ddot{p}_1 + bx_{1d} - ax_{2d}2ab],$$

$$\begin{aligned} p_2(t) = & \frac{a^2+b^2+\frac{1}{k}}{2ab}[\lambda e^{C_1 t} + \beta e^{-C_1 t} + \mu e^{C_2 t} + \alpha e^{-C_2 t} + \nu] - \frac{1}{2ab}[\lambda C_1^2 e^{C_1 t} + \beta C_1^2 e^{-C_1 t} + \mu C_2^2 e^{C_2 t} \\ + & \alpha C_2^2 e^{-C_2 t}] + \frac{1}{2a}x_{1d} - \frac{1}{2b}x_{2d}, \end{aligned}$$

$$\begin{aligned} p_2(t) = & \lambda[\frac{a^2+b^2+\frac{1}{k} - C_1^2}{2ab}]e^{C_1 t} + \beta[\frac{a^2+b^2+\frac{1}{k} - C_1^2}{2ab}]e^{-C_1 t} \\ + & \mu[\frac{a^2+b^2+\frac{1}{k} - C_2^2}{2ab}]e^{C_2 t} + \alpha[\frac{a^2+b^2+\frac{1}{k} - C_2^2}{2ab}]e^{-C_2 t} \\ + & (\frac{a^2+b^2+\frac{1}{k}}{2ab})\nu + \frac{1}{2a}x_{1d} - \frac{1}{2b}x_{2d}. \end{aligned}$$

Then,

$$p_2(t) = \lambda e^{C_1 t} + \beta e^{-C_1 t} - \mu e^{C_2 t} - \alpha e^{-C_2 t} + (\frac{a^2+b^2+\frac{1}{k}}{2ab})\nu + \frac{1}{2a}x_{1d} - \frac{1}{2b}x_{2d}. \tag{9}$$

Also, given the following equation:

$$x_1 = bp_1 - ap_2 - \dot{p}_1 + x_{1d},$$

$$x_2 = -ap_1 + bp_2 - \dot{p}_2 + x_{2d}.$$

We obtain the following results:

$$\begin{aligned} x_1(t) = & b[\lambda e^{C_1 t} + \beta e^{-C_1 t} + \mu e^{C_2 t} + \alpha e^{-C_2 t} + \nu] - a[\lambda e^{C_1 t} + \beta e^{-C_1 t} + \mu e^{C_2 t} + \alpha e^{-C_2 t} \\ + & \frac{a^2+b^2+\frac{1}{k}}{2ab}\nu + \frac{1}{2a}x_{1d} - \frac{1}{2b}x_{2d}] - [\lambda C_1 e^{C_1 t} - \beta C_1 e^{-C_1 t} + \mu C_2 e^{C_2 t} - \alpha C_2 e^{-C_2 t}] + x_{1d} \end{aligned}$$

$$\begin{aligned} x_1(t) &= \lambda(b-a-C_1)e^{C_1 t} + \beta(b-a+C_1)e^{-C_1 t} + \mu(b+a-C_2)e^{C_2 t} + \alpha(b+a+C_2)e^{-C_2 t} \\ &+ [b - \frac{(a^2+b^2+\frac{1}{k})}{2b}]\nu - \frac{1}{2}x_{1d} + \frac{a}{2b}x_{2d} + x_{1d}. \end{aligned}$$

Then,

$$\begin{aligned} x_1(t) &= \lambda(b-a-C_1)e^{C_1 t} + \beta(b-a+C_1)e^{-C_1 t} + \mu(b+a-C_2)e^{C_2 t} + \alpha(b+a+C_2)e^{-C_2 t} + \frac{b^2-a^2-\frac{1}{k}}{2b}\nu \\ &+ \frac{1}{2}x_{1d} + \frac{a}{2b}x_{2d}. \end{aligned} \quad (10)$$

$$\begin{aligned} x_2(t) &= -a[\lambda e^{C_1 t} + \beta e^{-C_1 t} + \mu e^{C_2 t} + \alpha e^{-C_2 t} + \nu] + b[\lambda e^{C_1 t} + \beta e^{-C_1 t} + \mu e^{C_2 t} + \alpha e^{-C_2 t} + \\ &+ (b^2 - a^2 - \frac{1}{k})2b]\nu + \frac{1}{2}x_{1d} + \frac{a}{2b}x_{2d} - [\lambda C_1 e^{C_1 t} - \beta C_2 e^{-C_1 t} + \mu C_2 e^{C_2 t} - \alpha C_2 e^{-C_2 t}] + x_{2d}, \end{aligned}$$

Then,

$$\begin{aligned} x_2(t) &= \lambda(b-a-C_1)e^{C_1 t} + \beta(b-a+C_1)e^{-C_1 t} + \mu(-b-a+C_2)e^{C_2 t} + \alpha(-a-b-C_2)e^{-C_2 t} \\ &+ \frac{b^2-a^2+\frac{1}{k}}{2a}\nu + \frac{b}{2a}x_{1d} + \frac{1}{2}x_{2d}. \end{aligned} \quad (11)$$

The constants are determined by the following limits conditions:
$x_1(0) = 0, x_2(0) = 0,$
$x_1(T) = 2,$
$p_1(T) = -k_1, p_2(T) = 0.$
We solve the linear system numerically:

$$\begin{cases} 0 = \lambda(b-a-C_1) + \beta(b-a+C_1) + \mu(b+a-C_2) + \alpha(b+a+C_2) + \frac{b^2-a^2-\frac{1}{k}}{2b}\nu + \frac{1}{2}x_{1d} + \frac{a}{2b}x_{2d}, \\ 0 = \lambda(b-a-C_1) + \beta(b-a+C_1) + \mu(-b-a+C_2) + \alpha(-b-a-C_2) + (\frac{b^2-a^2+\frac{1}{k}}{2a})\nu + \frac{b}{2a}x_{1d} + \frac{1}{2}x_{2d}, \\ 2 = \lambda(b-a-C_1)e^{C_1 T} + \beta(b-a+C_1)e^{-C_1 T} + \mu(b+a-C_2)e^{C_2 T} + \alpha(b+a+C_2)e^{-C_2 T} + \frac{b^2-a^2-\frac{1}{k}}{2b}\nu \\ +\frac{1}{2}x_{1d} + \frac{a}{2b}x_{2d}, \\ -k_1 = \lambda e^{C_1 T} + \beta e^{-C_1 T} + \mu e^{C_2 T} + \alpha e^{-C_2 T} + \nu, \\ 0 = \lambda e^{C_1 T} + \beta e^{-C_1 T} - \mu e^{C_2 T} - \alpha e^{-C_2 T} + (\frac{a^2+b^2+\frac{1}{k}}{2ab})\nu + \frac{1}{2a}x_{1d} - \frac{1}{2b}x_{2d} \end{cases} .$$

(12) is equivalent to following system:

$$\begin{cases} \lambda(b-a-C_1) + \beta(b-a+C_1) + \mu(b+a-C_2) + \alpha(b+a+C_2) = -\frac{b^2-a^2-\frac{1}{k}}{2b}\nu - \frac{1}{2}x_{1d} - \frac{a}{2b}x_{2d}, \\ \lambda(b-a-C_1) + \beta(b-a+C_1) + \mu(-b-a+C_2) + \alpha(-b-a-C_2) = -(\frac{b^2-a^2+\frac{1}{k}}{2a})\nu - \frac{b}{2a}x_{1d} - \frac{1}{2}x_{2d}, \\ \lambda(b-a-C_1)e^{C_1 T} + \beta(b-a+C_1)e^{-C_1 T} + \mu(b+a-C_2)e^{C_2 T} + \alpha(b+a+C_2)e^{-C_2 T} = 2 - \frac{b^2-a^2-\frac{1}{k}}{2b}\nu \\ -\frac{1}{2}x_{1d} - \frac{a}{2b}x_{2d}, \\ \lambda e^{C_1 T} + \beta e^{-C_1 T} + \mu e^{C_2 T} + \alpha e^{-C_2 T} + k_1 = -\nu, \\ \lambda e^{C_1 T} + \beta e^{-C_1 T} - \mu e^{C_2 T} - \alpha e^{-C_2 T} = -(\frac{a^2+b^2+\frac{1}{k}}{2ab})\nu - \frac{1}{2a}x_{1d} + \frac{1}{2b}x_{2d} \end{cases} .$$

Let matrix form:

$$\begin{pmatrix} b-a-C_1 & b-a+C_1 & b+a-C_2 & b+a+C_2 & 0 \\ b-a-C_1 & b-a+C_1 & -b-a+C_2 & -b-a-C_2 & 0 \\ (b-a-C_1)e^{C_1T} & (b-a+C_1)e^{-C_1T} & (-b-a+C_2)e^{C_2T} & (-b-a-C_2)e^{-C_2T} & 0 \\ e^{C_1T} & e^{-C_1T} & e^{C_2T} & e^{-C_2T} & 1 \\ e^{C_1T} & e^{-C_1T} & -e^{C_2T} & -e^{-C_2T} & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \beta \\ \mu \\ \alpha \\ k_1 \end{pmatrix} = \begin{pmatrix} -\frac{b^2-a^2-\frac{1}{k}}{2b}\nu - \frac{1}{2}x_{1d} - \frac{a}{2b}x_{2d} \\ -(\frac{b^2-a^2+\frac{1}{k}}{2a})\nu - \frac{b}{2a}x_{1d} - \frac{1}{2}x_{2d} \\ 2 - \frac{b^2-a^2-\frac{1}{k}}{2b}\nu - \frac{1}{2}x_{1d} - \frac{a}{2b}x_{2d} \\ -\nu \\ -(\frac{a^2+b^2+\frac{1}{k}}{2ab})\nu - \frac{1}{2a}x_{1d} + \frac{1}{2b}x_{2d} \end{pmatrix}$$

5.0.1 Comparison of the Two Approaches

The numerical algorithm is implemented in Matlab. Particularly, we used the function $ode45$ et $fsolve$. In this example, the method converge independently with initial point $\psi(0) = (0.0536, 0.1072)$. Numerical experience is determined by values $a = 1$, $b = 3$ et $k = 2$. Let us deduce that analytical solution and numerical solution correspond perfectly. For different values of k, The performance of the numerical procedure is

(a) (b)

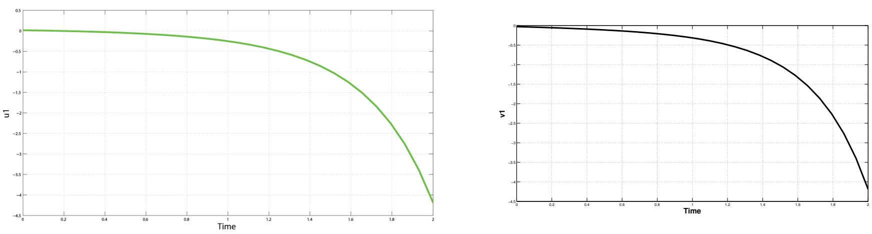

Figure 1: (a) u_1 numerical solution (b) v_1 true solution

(a) (b)

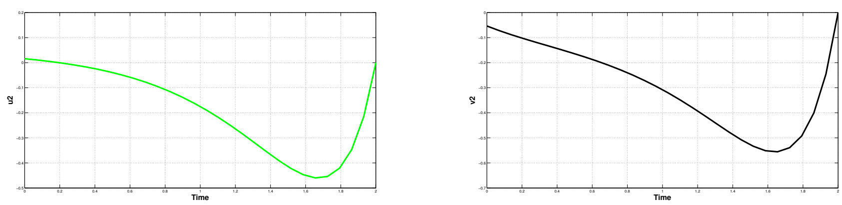

Figure 2: (a) u_2 numerical solution (b) v_2 true solution

summarized in the following table.

k	time C.P.U	Iteration Number
0.5	1.3089	12
1	2.3619	8
1.5	2.7016	7
2	3.7995	7
2.5	5.7479	6

Note that the algorithm converges quickly, to a very small number of iterations required to reach convergence. In addition, the computing time used is very low.

(a) (b)

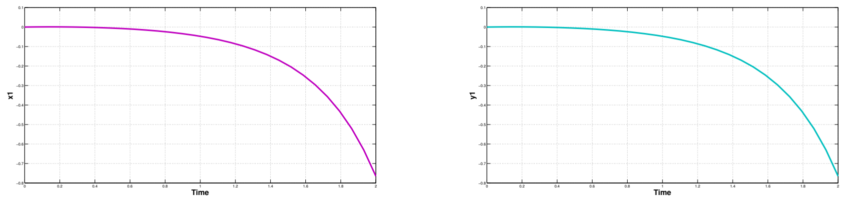

Figure 3: (a) x_1 numerical solution (b) y_1 true solution

(a) (b)

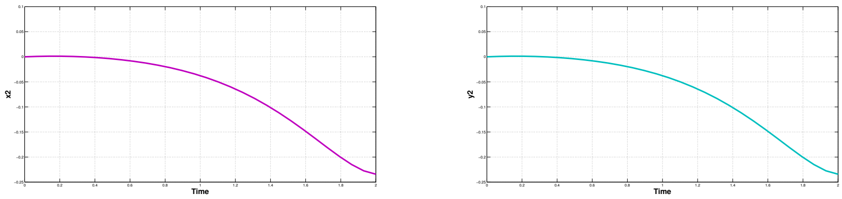

Figure 4: (a) x_2 numerical solution (b) y_2 true solution

(a) (b)

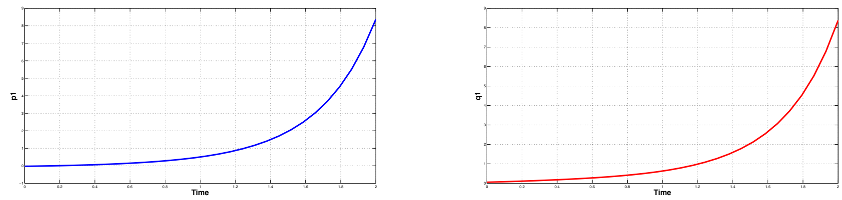

Figure 5: (a) p_1 numerical solution (b) q_1 true solution

(a) (b)

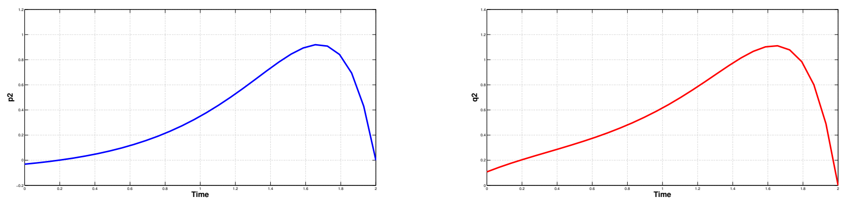

Figure 6: (a) p_2 numerical solution (b) q_2 true solution

6 Conclusions

The goal of this study concerns the linear quadratic optimal control problems with a fixed terminal time. We have used the relaxation method coupled with the shooting method for solving the optimal control problem. The convergence of the procedure is ensured. The rate of convergence is high and the computation of time is fast. We concluded that relaxation method give a good results in quick time.

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